J. Fluid Mech. (1978), *vol.* 87, *part* 2, *pp.* 335–341
Printed in Great Britain

Hydrodynamic pressures on sloping dams during earthquakes. Part 1. Momentum method

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(Received 5 October 1977 and in revised form 6 December 1977)

Von Kármán's momentum-balance method is adopted to investigate the earthquake forces on a rigid dam with an inclined upstream face of constant slope. The distribution of the hydrodynamic pressure along the upstream face is determined. It is found that the maximum hydrodynamic pressure occurs at the base of the dam for any inclination angle between 0 and 90°. Explicit analytical formulae for evaluating the total horizontal, vertical and normal loads are presented and a useful approximate rule for practical engineers is given.

1. Introduction

In the study of the earthquake response of rigid rectangular dams with vertical upstream faces, Westergaard (1933) first derived an expression for the hydrodynamic pressure exerted on a dam by an incompressible fluid in the reservoir as a result of horizontal harmonic ground motion in the direction perpendicular to the dam. He found that the hydrodynamic pressure was the same as if a certain body of fluid was forced to move back and forth with the dam and that this 'added mass' was confined in a volume bounded by a two-dimensional parabolic surface on the upstream side of the dam. On the basis of a simple linear momentum-balance principle, von Kármán (1933) obtained distributions of the hydrodynamic pressure force and the total load on a rigid dam with a vertical upstream face which were very close to the Westergaard results.

For a dam whose upstream face is not vertical, Zangar (1953) and Zangar & Haefeli (1952) determined the hydrodynamic pressures experimentally using an electrical analogue. Because of the mathematical difficulties involved in determining exactly the hydrodynamic pressure exerted on an inclined upstream face of a dam by horizontal earthquakes, no analytical solutions are available in the literature. The objective of this paper is to deduce an analytical solution for the earthquake force on a rigid sloping dam by adopting the von Kármán momentum-balance approach. From this approach, we are able to deduce the distribution of the hydrodynamic pressure (excluding the hydrostatic pressure) exerted along the inclined upstream face of the dam by horizontal earthquake motions. Explicit analytic formulae for calculating the total horizontal, vertical and normal loads are also presented.

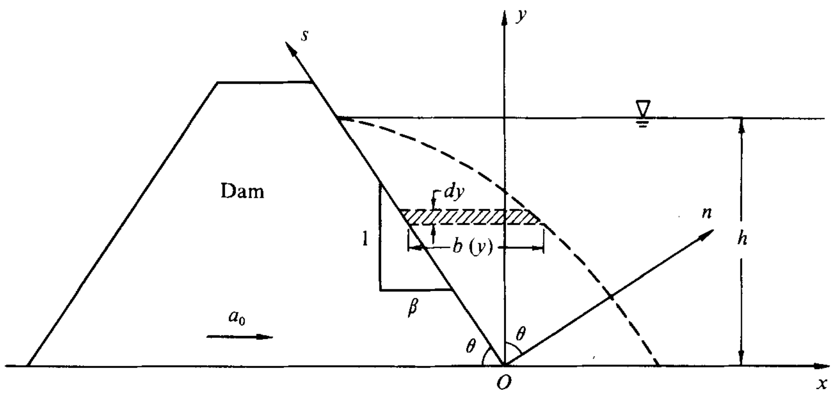

FIGURE 1. Schematic diagram of a dam with an inclined upstream face of constant slope.

2. Governing equations

Let us consider a dam with an inclined upstream face of constant slope $1/\beta$ (see figure 1). If the angle between the upstream face and the $-x$ axis is θ, then

$$\beta = \cot\theta. \tag{1}$$

The y axis points vertically upwards and the still water level in the reservoir is at $y = h$, the plane $y = 0$ being the bottom of the reservoir. The dam, which is at rest at time $t = 0$, is assumed to have a uniform horizontal acceleration a_0 in the x direction during the time interval Δt. Because of this accelerated motion of the dam, a portion of the fluid will be set in motion. As a result, the fluid will exert a hydrodynamic pressure on the upstream face of the dam that is proportional to a_0, in addition to the hydrostatic pressure. This effect has been called the 'apparent-mass' or 'added-mass' effect.

To investigate the added-mass effect, we adopt a generalized version of von Kármán's (1933) momentum-balance approach. Let the s axis be tangential to the upstream face of the dam in the x,y plane and let the n axis be perpendicular to the face of the dam (see figure 1). The fluid in the reservoir is taken to be incompressible and inviscid. Since the dam is assumed to be rigid, the normal component a_{0n} (in the n direction) of the acceleration is

$$a_{0n} = a_0 \sin\theta = a_0(1+\beta^2)^{-\frac{1}{2}}, \tag{2}$$

which is a constant. For an inviscid fluid, the only boundary condition at the dam–fluid interface S_d is that the normal velocity of fluid particles must be the same as that of the dam at all times. Hence

$$a_n(x,y) = a_{0n} \quad \text{on} \quad x = -\beta y. \tag{3}$$

We shall assume that, owing to the horizontal acceleration a_0 of the dam, an effective mass of fluid of width $b(y)$ experiences the full value of the normal acceleration a_{0n} in the n direction, while the remainder of the fluid is not involved. Alternatively, we may define $b(y)$ by the equation

$$\int_{-\beta y}^{\infty} a_n(x,y)\,dx = b a_{0n} \quad (0 \leqslant y \leqslant h). \tag{4}$$

If we denote the components of the fluid acceleration in the x, y and s directions by a_x, a_y and a_s respectively, then we have

$$a_x(x,y) = a_{0n}\sin\theta - a_s(x,y)\cos\theta, \tag{5}$$

$$a_y(x,y) = a_{0n}\cos\theta + a_s(x,y)\sin\theta. \tag{6}$$

Following von Kármán's approach, the continuity condition that the fluid mass displaced by the accelerated motion of the dam between $y = 0$ and an arbitrary height y must pass through the section $b(y)$ requires that

$$ya_0 = ba_{0n}\cos\theta + f\sin\theta, \tag{7}$$

where f is defined by

$$f(y) = \int_{-\beta y}^{b-\beta y} a_s(x,y)\,dx. \tag{8}$$

The balance of the linear momentum in the x direction for the mass of fluid of width $b(y)$ and height dy (see figure 1) requires that

$$p = \rho(ba_{0n}\sin\theta - f\cos\theta), \tag{9}$$

where p is the hydrodynamic pressure and ρ the constant density of the fluid. The balance of the linear momentum of the fluid in the y direction requires that

$$p\cot\theta - d(pb)/dy = \rho(ba_{0n}\cos\theta + f\sin\theta). \tag{10}$$

From (1), (2), (7) and (9), we have

$$p = \rho a_0(b - \beta y). \tag{11}$$

Substituting (7) and (11) into (10), we obtain the governing differential equation for the width $b(y)$ of the added mass as

$$\beta(b-\beta y) - d[b(b-\beta y)]/dy = y. \tag{12}$$

The pressure p vanishes at the fluid surface so the boundary condition for b at $y = h$ is

$$b = \beta h \quad \text{at} \quad y = h. \tag{13}$$

3. Pressure distribution and total load

By introducing a new variable

$$A(y) \equiv 2b - \beta y, \tag{14}$$

we may reduce (12) to

$$A\,dA/dy - \beta A = -2y. \tag{15}$$

The boundary condition for A corresponding to (13) is

$$A = \beta h \quad \text{at} \quad y = h. \tag{16}$$

The solution to (15) satisfying the boundary condition (16) is

$$\log_e\left(\frac{A^2 - \beta Ay + 2y^2}{2h^2}\right)$$

$$= \begin{cases} \dfrac{2\beta}{(8-\beta^2)^{\frac{1}{2}}}\left[\tan^{-1}\left(\dfrac{\beta}{(8-\beta^2)^{\frac{1}{2}}}\right) - \tan^{-1}\left(\dfrac{2A-\beta y}{y(8-\beta^2)^{\frac{1}{2}}}\right)\right] \quad (\beta^2 < 8), & (17a) \\[2ex] \dfrac{\beta}{(\beta^2-8)^{\frac{1}{2}}}\left[\log_e\left(\dfrac{\beta-(\beta^2-8)^{\frac{1}{2}}}{\beta+(\beta^2-8)^{\frac{1}{2}}}\right) - \log_e\left(\dfrac{2A-\beta y-(\beta^2-8)^{\frac{1}{2}}y}{2A-\beta y+(\beta^2-8)^{\frac{1}{2}}y}\right)\right] \quad (\beta^2 > 8). & (17b) \end{cases}$$

Equation (17) determines the variable $A(y)$ and, through (14), the shape $b(y)$ of the added fluid mass as implicit functions of y. In particular, for $y = 0$, (17) gives an explicit expression for b_0, the value of b at $y = 0$:

$$\frac{b_0}{h} = \begin{cases} \dfrac{1}{2^{\frac{1}{2}}}\exp\left\{-\dfrac{\beta}{(8-\beta^2)^{\frac{1}{2}}}\left[\dfrac{\pi}{2}-\tan^{-1}\left(\dfrac{\beta}{(8-\beta^2)^{\frac{1}{2}}}\right)\right]\right\} & (\beta^2 < 8), \quad (18a)\\[2ex] \dfrac{1}{2^{\frac{1}{2}}}\left[\dfrac{\beta-(\beta^2-8)^{\frac{1}{2}}}{\beta+(\beta^2-8)^{\frac{1}{2}}}\right]^{\beta/2(\beta^2-8)^{\frac{1}{2}}} & (\beta^2 > 8). \quad (18b)\end{cases}$$

For small values of β, (18a) becomes

$$\frac{b_0}{h} = \frac{1}{2^{\frac{1}{2}}}\left(1-\frac{\pi}{4\times 2^{\frac{1}{2}}}\beta+\frac{\pi^2+8}{64}\beta^2-\dots\right) \quad (\beta \ll 1), \tag{19}$$

which further reduces to the well-known von Kármán result for $\beta = 0$:

$$b_0/h = 1/2^{\frac{1}{2}} = 0{\cdot}707 \quad (\beta = 0). \tag{20}$$

As β approaches infinity, (18b) indicates that

$$\lim_{\beta\to\infty} b_0/h = 0. \tag{21}$$

When $\beta^2 = 8$, (18a) and (18b) yield the same limiting value for b_0:

$$\lim_{\beta^2\to 8}\frac{b_0}{h} = \frac{1}{2^{\frac{1}{2}}e} = 0{\cdot}260. \tag{22}$$

The pressure coefficient C_p, defined by

$$p = C_p\rho a_0 h, \tag{23}$$

is related to $A(y)$ through (11) and (14) by

$$C_p = (A-\beta y)/2h. \tag{24}$$

The pressure distribution along the upstream face of the dam is shown in figure 2 for several values of θ; these results were determined by applying the Newton–Raphson method to the nonlinear algebraic relation (17). In particular, when $\theta = 90°$ (i.e. $\beta = 0$), (17a) and (24) give

$$C_p = \frac{1}{2^{\frac{1}{2}}}\left[1-\left(\frac{y}{h}\right)^2\right]^{\frac{1}{2}} \quad (\beta = 0). \tag{25}$$

It can be noted from figure 2 that, for a fixed height y/h, the pressure decreases as the angle θ decreases, while for a fixed value of θ it increases as the height decreases and attains a maximum value of C_{p_0} at the bottom of the reservoir $y = 0$, where

$$C_{p_0} = b_0/h \tag{26}$$

with b_0/h given by (18). The fact that the hydrodynamic pressure reaches its maximum at $y = 0$ for fixed θ (or fixed β) can be verified easily from (17) and (24) since $dA/dy = \beta$, $dC_p/dy = 0$ and $d^2C_p/dy^2 < 0$ there.

The total vertical load on the sloping dam can be found by integrating (11) as

$$F_y = \int_0^h p\left(\frac{dy}{\sin\theta}\right)\cos\theta = \beta\rho a_0\int_0^h (b-\beta y)\,dy. \tag{27}$$

However, (12) gives

$$\beta\int_0^h (b-\beta y)\,dy = \tfrac{1}{2}h^2-b_0^2. \tag{28}$$

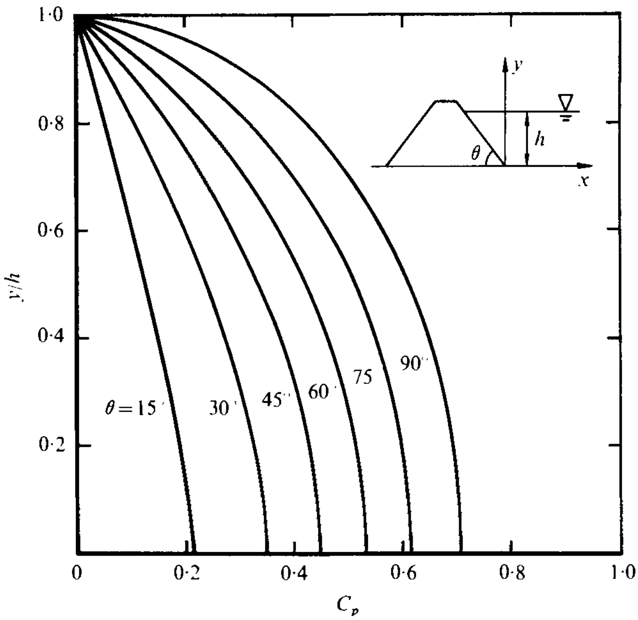

FIGURE 2. The pressure distribution on the upstream face of a dam.

In deriving (28), the boundary condition (13) has been applied. Therefore by means of (28), (27) may be simplified to

$$F_y = C_y \rho a_0 h^2, \quad C_y = \tfrac{1}{2} - (b_0/h)^2, \tag{29}$$

where b_0/h is given by (18).

Similarly, the total horizontal load on the upstream face of the dam is given by

$$F_x = C_x \rho a_0 h^2, \quad C_x = C_y/\beta, \tag{30}$$

and the total normal load is

$$F_n = C_n \rho a_0 h^2, \quad C_n = (C_x^2 + C_y^2)^{\frac{1}{2}}. \tag{31}$$

The force coefficients C_x, C_y and C_n as well as the maximum pressure coefficient C_{p_0}, given by (30), (29), (31) and (26) respectively, are plotted in figure 3 *vs.* the angle θ or the reciprocal of the slope β. When the upstream face of the dam becomes vertical, we have from (29), (30), (31) and (18) that

$$C_x = C_n = \pi/(4 \times 2^{\frac{1}{2}}) = 0{\cdot}555, \quad C_y = 0 \quad (\beta = 0), \tag{32}$$

which agrees exactly with the von Kármán result. When the upstream face becomes almost horizontal, we obtain

$$C_x = 0, \quad C_y = C_n = \tfrac{1}{2} \quad (\beta \to \infty), \tag{33}$$

which is expected by physical intuition. An interesting feature exhibited by figure 3 is that the total normal force remains almost constant over the whole range

$$0 \leqslant \theta \leqslant 90^\circ$$

of the inclination angle. The approximation

$$C_n \doteqdot 0{\cdot}5 \quad (0 \leqslant \theta \leqslant 90^\circ) \tag{34}$$

could be useful to dam engineers for making quick estimates.

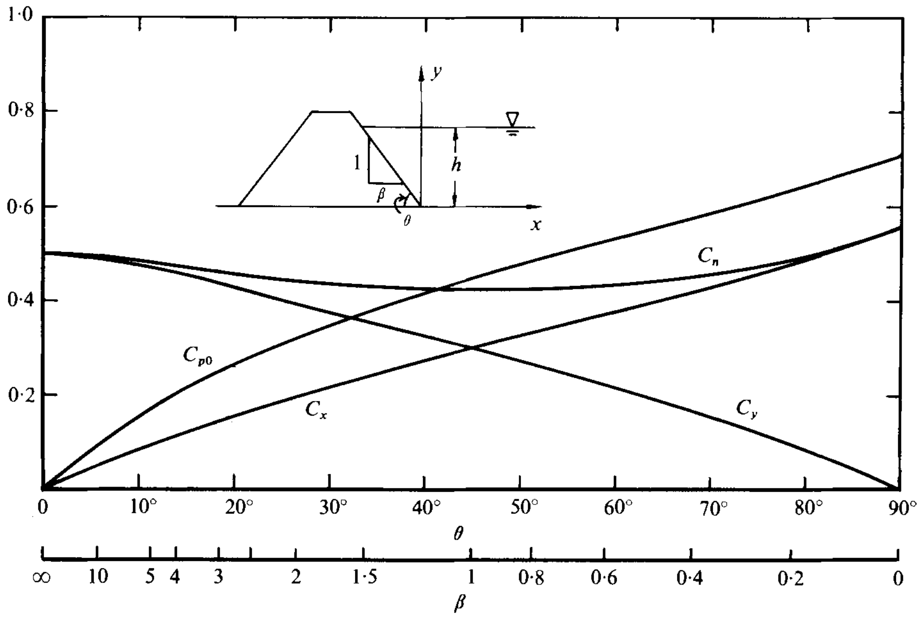

FIGURE 3. The horizontal, vertical and normal force coefficients and the maximum pressure coefficient *vs.* the inclination angle θ or the reciprocal of the slope β.

The pressure distributions given by the foregoing analytical solution for the sloping face differ somewhat from those given by Zangar. His 'experimental solution' indicates a peak pressure somewhat above $y = 0$, but his 'empirical solution' has peak pressure at $y = 0$. His 'empirical solution' for F_x is about 1 to 2 % smaller than our analytical solution, while his 'experimental solution' for F_x is from 1 to 8 % smaller than our analytical solution, greater deviation occurring at smaller angles θ.

4. Conclusions

The problem of the added-mass effect of horizontal acceleration of a dam with an inclined upstream face of constant slope has been solved analytically by von Kármán's momentum-balance approach. It has been found that the hydrodynamic pressure decreases as the slope decreases for any fixed height, while for a fixed slope it increases with depth beneath the water surface and always attains a maximum value at the bottom of the reservoir. Explicit analytical formulae for evaluating the total horizontal, vertical and normal forces have been given. The present solution has been found to agree exactly with the von Kármán result when the upstream face becomes vertical. An approximate rule based on the present result which may be useful to dam engineers states that the normal force coefficient remains practically constant at around 0·5 for all slopes.

The authors are indebted to Professor Ronald F. Scott for valuable suggestions and comments. This work was partially sponsored by the National Science Foundation, under Grant PFR 77-16085.

REFERENCES

KÁRMÁN, T. VON 1933 Discussion of water pressures on dams during earthquakes. *Trans. A.S.C.E.* **98**, 434–436.

WESTERGAARD, H. M. 1933 Water pressures on dams during earthquakes. *Trans. A.S.C.E.* **98**, 418–433.

ZANGAR, C. N. 1953 Hydrodynamic pressures on dams due to horizontal earthquakes. *Proc. Soc. Exp. Stress Anal.* **10**, 93–102.

ZANGAR, C. N. & HAEFELI, R. J. 1952 Electric analog indicates effect of horizontal earthquake shock on dams. *Civil Engng* **22**, 278–279.